Week 2 - Friday

COMP 2230

Last time

- Predicate logic
  - Universal quantifier
  - Existential quantifier
  - Negating quantifiers
  - Multiple quantifiers

Questions?

Assignment 1

Logical warmup

- A bat costs $1 more than a ball
- Together, they cost $1.10
- How much does the ball cost?

Multiple Quantifiers

Practice

- Given the formal statements with multiple quantifiers for each of the following:
  - There is someone for everyone.
  - All roads lead to some city.
  - Someone in this class is smarter than everyone else.
  - There is no largest prime number.

Negating multiply quantified statements

- The rules don't change
- Simply switch every ∀ to ∃ and every ∃ to ∀
- Then negate the predicate
- Write the following formally:
  - "Every rose has a thorn"
- Now, negate the formal version
- Convert the formal version back to informal

Changing quantifier order

- As show before, changing the order of quantifiers can change the truth of the whole statement
- However, it does not necessarily
- Furthermore, quantifiers of the same type are commutative:
  - You can reorder a sequence of ∀ quantifiers however you want
  - The same goes for ∃
  - Once they start overlapping, however, you can't be sure anymore

Arguments with Quantified Statements

Quantification in arguments

- Quantification adds new features to an argument
- The most fundamental is **universal instantiation**
  - If a property is true for everything in a domain (universal quantifier), it is true for any specific thing in the domain
- Example:
  - All the party people in the place to be are throwing their hands in the air!
  - Julio is a party person in the place to be
  - $\therefore$ Julio is throwing his hands in the air

Universal modus ponens

- Formally,
  - $\forall x, P(x) \rightarrow Q(x)$
  - $P(a)$ for some particular a
  - $\therefore Q(a)$
- Example:
  - If any person disses Dr. Dre, he or she disses him or herself
  - Tammy disses Dr. Dre
  - Therefore, Tammy disses herself

Universal modus tollens

- Much the same as universal modus ponens
- Formally,
  - $\forall x, P(x) \rightarrow Q(x)$
  - $\sim Q(a)$ for some particular a
  - $\therefore \sim P(a)$
- Example:
  - Every true DJ can skratch
  - John Comerford can't skratch
  - Therefore, John Comerford is not a true DJ

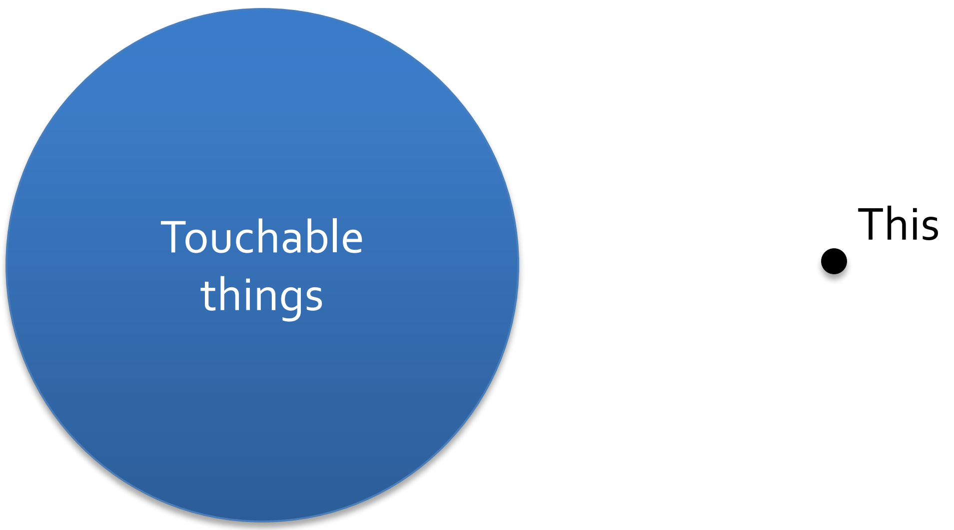
Inverse and converse errors strike again

- Unsurprisingly, the inverse and the converse of universal conditional statements do not have the same truth value as the original
- Thus, the following are not valid arguments:
  - If a person is a superhero, he or she can fly.
  - Astronaut John Blaha can fly.
  - Therefore, John Blaha is a superhero. **FALLACY**

  - A good man is hard to find.
  - Osama Bin Laden is not a good man.
  - Therefore, Osama Bin Laden is not hard to find. **FALLACY**

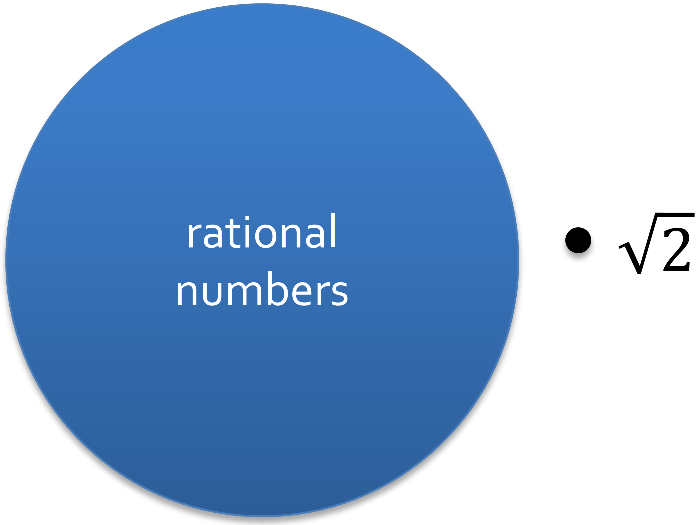
Venn diagrams

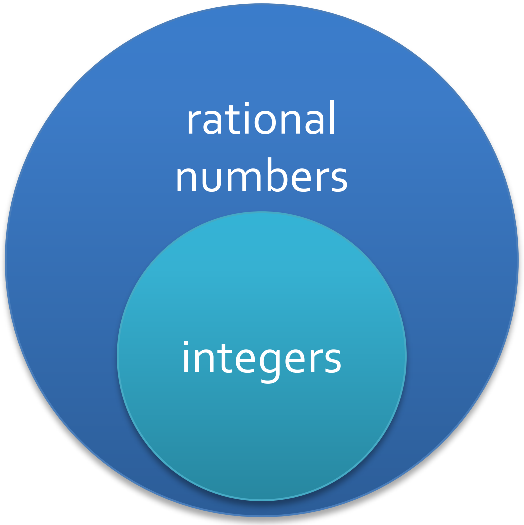
- We can test arguments using Venn diagrams
- To do so, we draw diagrams for each premise and then try to combine the diagrams

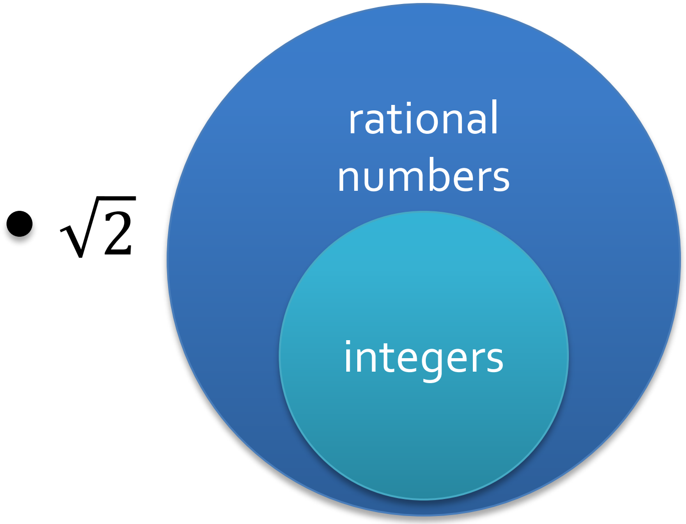


Diagrams showing validity

- All integers are rational numbers
- $\sqrt{2}$ is not rational







- Therefore, $\sqrt{2}$ is not an integer

Diagrams showing invalidity

- All tigers are cats

- Panthro is a cat

- Therefore, Panthro is a tiger

Be careful

- Diagrams can be useful tools
- However, they don't offer the guarantees that pure logic does
- Note that the previous slide makes the converse error unless you are very careful with your diagrams

Two kinds of equal signs

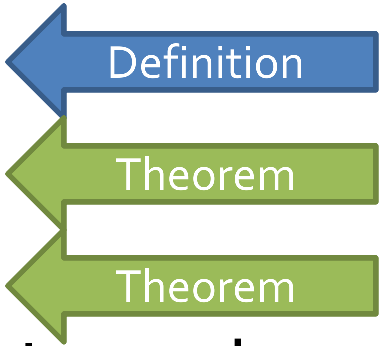
- In Java (and many other languages), there are two kinds of equal signs: **=** (assignment) and **==** (testing for equality)
- Mathematicians have two as well, but they use the same symbol!
- One kind of equal sign is a definition
- The other kind is a theorem

Definition equals

- The equal sign used for a definition is stating some fact, often defining what words mean
- Example:
  - $x^2 = x \cdot x$
  - It's not like some argument was needed to show that x^2 means $x \cdot x$, it's the definition!
- Sometimes bidirectional implication is used in this situation, arguably in a more clear way:
  - x is even $\leftrightarrow$ $x = 2k$, for some integer k
- Some people will use $\equiv$ or $\stackrel{\text{def}}{=}$

Theorem equals

- The other kind of equals shows a fact that has been derived from other facts
  - We've discovered that it's true!
- Example:
  - $3x + 2 = 11$
  - $3x = 9$
  - $x = 3$



- Sometimes confusion arises when people mistake one kind of equals for another

Proving Existential Statements and Disproving Universal Ones

A useful definition

- We'll start with basic definitions of even and odd to allow us to prove simpler theorems
- If n is an integer, then:
  - n is even $\Leftrightarrow \exists\, k \in \mathbb{Z}, n = 2k$
  - n is odd $\Leftrightarrow \exists\, k \in \mathbb{Z}, n = 2k + 1$
- Since these are bidirectional, each side implies the other

Another useful definition

- If n is an integer where $n > 1$, then:
  - n is prime $\Leftrightarrow \forall\, r \in \mathbb{Z}^+, \forall\, s \in \mathbb{Z}^+$, if $n = r \cdot s$, then $r = 1$ or $s = 1$
  - n is composite $\Leftrightarrow \exists\, r \in \mathbb{Z}^+, \exists\, s \in \mathbb{Z}^+, n = r \cdot s$ and $r \neq 1$ and $s \neq 1$

Proving existential statements

- A statement like the following:

  $\exists x \in D, P(x)$

- is true, if and only if, you can find at least one element of D that makes $P(x)$ true
- To prove this, you either have to find such an x or give a set of steps to find one
- Doing so is called a **constructive proof of existence**
- There are also **nonconstructive proofs of existence** that depend on using some other axiom or theorem

Examples

- Prove that there is a positive integer that can be written as the sum of the squares of two positive integers in two distinct ways
- More formally, prove:
  - $\exists x, y, z, a, b \in \mathbb{Z}^{+}, x = y^2 + z^2$ and $x = a^2 + b^2$ and $y \neq a$ and $y \neq b$
- Suppose that r and s are integers. Prove that there is an integer k such that $22r + 18s = 2k$

Disproving universal statements

- Disproving universal statements is structurally similar to proving existential ones
- Instead of needing any single example that works, we need a single example that doesn't work, called a **counterexample**
- Why?
- To disprove $\forall x \in D, P(x) \rightarrow Q(x)$, we need to find an x that makes $P(x)$ true and $Q(x)$ false

Examples

- Using counterexamples, disprove the following statements:
- $\forall a, b \in \mathbb{R}$, if $a^2 = b^2$ then $a = b$
- $\forall x \in \mathbb{Z}^+$, if $x \geq 2$ and x is odd, x is prime
- $\forall y \in \mathbb{Z}^+$, if y is odd, then $(y - 1)/2$ is prime

Proving Universal Statements

Method of exhaustion

- If the domain is finite, try every possible value.
- Example:
  - $\forall x \in \mathbb{Z}^+$, if $4 \le x \le 10$ and x is even, x can be written as the sum of two prime numbers
- Is this familiar to anyone?
- Goldbach's Conjecture proposes that this is true for all even integers greater than 2

Generalizing from the generic particular

- Pick some specific (but arbitrary) element from the domain
- Show that the property holds for that element, just because of that properties that any such element must have
- Thus, it must be true for all elements with the property
- Example: $\forall x \in \mathbb{Z}$, if x is even, then $x + 1$ is odd

Direct proof

- Direct proof actually uses the method of generalizing from a generic particular, following these steps:
  1. Express the statement to be proved in the form $\forall x \in D$, if $P(x)$ then $Q(x)$
  2. Suppose that x is some specific (but arbitrarily chosen) element of D for which $P(x)$ is true
  3. Show that the conclusion $Q(x)$ is true by using definitions, other theorems, and the rules for logical inference

Proof formatting

- Write the statement of the theorem
- Start your proof with the word **Proof**
- Define everything
- Write a justification next to every line
- Put a ■ or a **QED** at the end of your proof
  - Quod erat demonstrandum: "that which was to be shown"

Direct proof example

- Prove the sum of any two odd integers is even.

Common mistakes

- Arguing from examples
  - Goldbach's conjecture is not a proof, though shown for numbers up to 10^{18}
- Using the same letter to mean two different things
  - $m = 2k + 1$ and $n = 2k + 1$
- Jumping to a conclusion
  - Skipping steps
- Begging the question
  - Assuming the conclusion
- Misuse of the word if
  - A more minor problem, but a premise should not be invoked with "if"

Disproving an existential statement

- Flipmode is the squad
- You negate the statement and then prove the resulting universal statement

Upcoming

Next time…

- **No class Monday!**
- More proofs
  - Rational numbers
  - Divisibility
  - Proof by cases

Reminders

- Read Sections 4.3, 4.4, 4.5, and 4.6
- Keep working on Assignment 1